

(iii)

Hence $N(\epsilon) = N \left(\frac{T}{T_B} \right)^{3/2}$

From

$$N = n_0 + N(\epsilon)$$

$$N = n_0 + N \left(\frac{T}{T_B} \right)^{3/2}$$

Get g/sate population (condensation)

$$n_0 = N \left[1 - \left(\frac{T}{T_B} \right)^{3/2} \right]$$

Graphs of n_0 vs T } see sheet.
 B vs T }

Points

In 'condensation' temp range $0 < T \leq T_B$

- (i) G/sate occupation falls from N at $T=0$
to 0 at $T=T_B$

Many (n_0) particles carrying no energy —
do not contribute to U , C_V , S

Condensed fraction in absolute zero state.

(112)

(ii) Total energy U given by bridge equation

$$\frac{U}{N(T)} = \frac{\partial}{\partial \beta} (\ln Z)$$

$$\text{where } Z = V \left(\frac{2\pi m k T}{h^2} \right)^{3/2} = \text{const. } T^{3/2}$$

$$U = N(T) \cdot k T^2 \frac{\partial}{\partial T} (\ln (\text{const. } T^{3/2}))$$

$$U = N \left(\frac{T}{T_B} \right)^{3/2} k T^2 \cdot \frac{3}{2} \cdot \frac{1}{T}$$

$$U = \frac{3}{2} \frac{Nk}{T_B^{3/2}} \cdot T^{5/2}$$

$$(iii) \text{ Heat Capacity } C_V = \left(\frac{\partial U}{\partial T} \right)_V$$

$$C_V = \frac{15}{4} Nk \left(\frac{T}{T_B} \right)^{3/2}$$

(iv) Pressure

$$P = \frac{2U}{3V}$$

— see below.

(112)

(ii) Total energy U given by bridge equation

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(iii) Heat Capacity $C_V = \left(\frac{\partial U}{\partial T} \right)_V$

$$C_V = \frac{15}{4} Nk \left(\frac{T}{T_B} \right)^{3/2}$$

(iv) Pressure

$$P = \frac{2U}{3V}$$

— see below.

$$\text{Thus } p = \frac{Nk}{V} \frac{T^{5/2}}{T_B^{3/2}}$$

These variations of U , C_v , P are for temperature region $0 < T < T_B$.

(V) In temperature region $T \gg T_B$ again classical relations

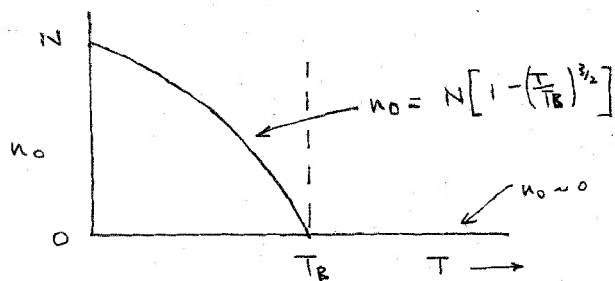
$$U = \frac{3}{2} NkT$$

$$C_v = \frac{3}{2} Nk$$

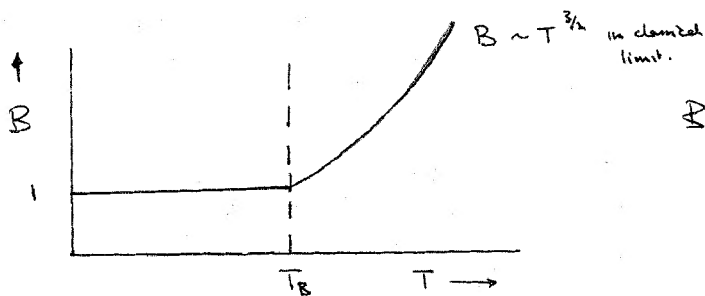
$$p = \frac{NkT}{V}$$

Variation of $\begin{matrix} U \\ C_v \\ p \end{matrix}$ $\left\{ \begin{matrix} \text{vs } T \\ - \text{ see sheet.} \end{matrix} \right.$

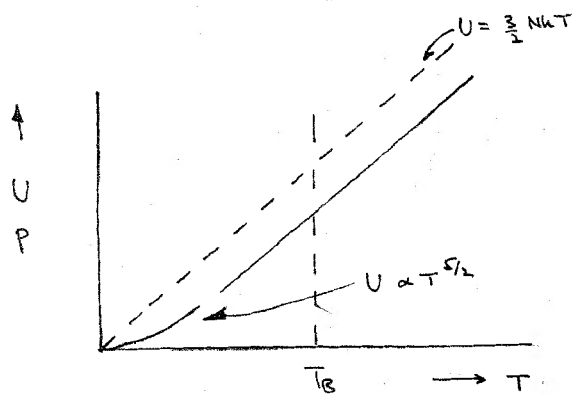
Boson particle gas.



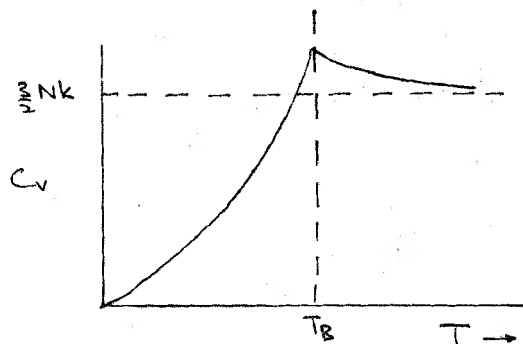
N_0 vs T



B vs T



U vs T



C_V vs T

Bose - Einstein condensation causes dramatic effects in properties of

- (i) Liquid He^4
- (ii) Superconducting materials

These discussed in Low Temperature part of course.

(115)

Massless boson systems

Photons — electromagnetic radiation

Phonons — quantised lattice waves in solids.

Photons.

Radiation in an enclosure (black box)

Want to derive the spectrum of radiation that is in equilibrium with enclosure of temperature T .

Spectrum is distribution $u(\nu)d\nu$ — the energy contained in radiation of frequency between ν and $\nu + d\nu$.

Want $u(\nu)$ as function of ν

Planck's radiation formula.

116

Treat photons in an enclosure as
Bose - Einstein gas.

but N not constant — photons
can be created / destroyed at walls.

Follow same B-E derivation but
omit condition $\sum_i dn_i = 0$

so in previous case put $\alpha = 0$.

$$\text{Get } f(\epsilon) = \frac{1}{[\exp(\epsilon/kT) - 1]}$$

Derivation of $u(\nu)$

Photons are waves in a box.

Number of states with $k \rightarrow k+dk$ is

$$g(k) dk = \frac{V}{(2\pi)^3} \cdot 4\pi k^2 dk \times 2$$

↖ 2
polarisations.
↑ and ↔

(117) Want distribution in terms of freq ν

Relation $\nu = \frac{c}{\lambda} = \frac{kc}{2\pi}$

$$k^2 dk = \left(\frac{2\pi}{c}\right)^3 \nu^2 d\nu$$

Number of states in range $\nu \rightarrow \nu + d\nu$

$$g(\nu) d\nu = \frac{V}{(2\pi)^3} \cdot \left(\frac{2\pi}{c}\right)^3 \cdot 8\pi \nu^2 d\nu$$

$$= \frac{8\pi V}{c^3} \nu^2 d\nu$$

Energy distribution

$$u(\nu) d\nu = g(\nu) d\nu \cdot f(\nu) \cdot h\nu$$

$$= \frac{8\pi V}{c^3} \nu^2 d\nu \cdot \frac{1}{[\exp(\frac{h\nu}{kT}) - 1]} \cdot h\nu$$

$$u(\nu) d\nu = \frac{8\pi V h \nu^3 d\nu}{c^3 [\exp(\frac{h\nu}{kT}) - 1]}$$

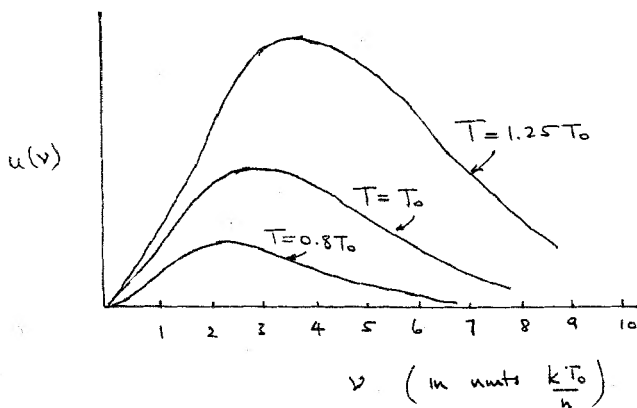
Planck radiation formula for black body

Plot - see

Planck radiation formula.

In black body enclosure at temperature T -
distribution of energy u as function of photon
frequency ν is

$$u(\nu) = \frac{8\pi V h \cdot \nu^3}{c^3 [\exp(\frac{h\nu}{kT}) - 1]}$$



Planck formula for

$0.8T_0$
T_0
$1.25T_0$

Area under curve increases as T^4 .

(118)

Points

- (i) Excellent agreement with expr for $u(\nu)$.
- (ii) For maxima in distributions at ν_{\max} —
 $\nu_{\max} \propto T$. (Wien's law).
- (iii) Total energy density $\left(\frac{U}{V}\right)$ is integral under curve.

$$\frac{U}{V} = \int_0^{\infty} u(\nu) d\nu$$

$$\frac{U}{V} = \frac{8\pi h}{c^3} \int_0^{\infty} \frac{\nu^3 d\nu}{\left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]}$$

Putting $y = \frac{h\nu}{kT}$ $d\nu = \frac{kT}{h} dy$

$$\begin{aligned} \frac{U}{V} &= \frac{8\pi h}{c^3} \left(\frac{kT}{h}\right)^4 \int_0^{\infty} \frac{y^3 dy}{\left[\exp(y) - 1\right]} \\ &= \frac{8\pi h}{c^3} \left(\frac{kT}{h}\right)^4 \left(\frac{\pi^4}{15}\right) \leftarrow \text{integral} \end{aligned}$$

119

$$\frac{U}{V} = \frac{8\pi^5}{15} \left(\frac{1}{hc} \right)^3 (kT)^4.$$

(iv) Stefan's Law.

Energy radiated $\text{m}^2 \text{s}^{-1}$ from black body $= \sigma T^4$.

Number of photons emitted $/\text{s} \cdot \text{m}^2 = \frac{1}{4} \frac{N}{V} \cdot c$

— see sheet.

$$\text{Energy emitted } / \text{m}^2 \text{s} = \frac{1}{4} \left(\frac{N}{V} \right) c \cdot \overline{h\nu}$$

where $\overline{h\nu}$ is mean photon energy

$$\text{Energy emitted } / \text{m}^2 \text{s} = \frac{1}{4} \left(\frac{U}{V} \right) \cdot c$$

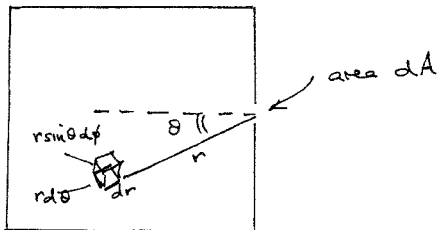
$$= \frac{1}{4} \cdot c \left(\frac{8\pi^5 h}{15 c^3} \right) \frac{\pi^4}{15} \cdot \left(\frac{kT}{h} \right)^4$$

Excellent agreement with exp^t for

(i) variation as T^4

(ii) Value of σ

Number of photons striking area 1 m^2 / second.



Consider elemental box of side $dr, r d\theta, r \sin\theta d\phi$ as shown.

Number of photons in this volume $= \left(\frac{N}{V}\right) \cdot r^2 \sin\theta dr d\theta d\phi$.

Fraction that will go thru $dA = \frac{1}{4\pi} \cdot \frac{dA \cdot \cos\theta}{r^2}$

Number of photons from box passing thru dA

$$= \left(\frac{N}{V}\right) \cdot \frac{1}{4\pi} \cdot \frac{dA \cos\theta}{r^2} \cdot r^2 \sin\theta dr d\theta d\phi$$

Total number of photons passing thru dA/s .

$$= \frac{dA}{4\pi} \cdot \left(\frac{N}{V}\right) \cdot \int_0^c dr \int_0^{\pi/2} \cos\theta \sin\theta d\theta \int_0^{2\pi} d\phi$$

$$= \frac{dA}{4\pi} \cdot \left(\frac{N}{V}\right) \cdot c \cdot \frac{1}{2} \cdot 2\pi$$

$$= dA \cdot \frac{1}{4} \left(\frac{N}{V}\right) c$$

Number photons hitting $1 \text{ m}^2/s = \frac{1}{4} \left(\frac{N}{V}\right) c$.

(120)

(v) Radiation pressure

$$P = \frac{U}{3V} \quad - \text{see sheet.}$$

(vi) Photon theory - valid over what range of temperature?

Integration must be valid

Requires $kT > \Delta E$ - (energy state spacing)

For states in box of side a

$$\Delta E \sim \frac{hc}{a}$$

Limiting value of temperature T where

$$kT = \frac{hc}{a}$$

$$T = \frac{hc}{ka} \sim 1 \text{ K for } a = 0.01 \text{ m.}$$

At temperature $T = 1 \text{ K}$ radiation not important in practice - theory ok.

Pressures.

Radiation pressure and pressure from particles with mass.

For both
$$p = - \left(\frac{\partial U}{\partial V} \right)_S$$

$$p = - \frac{\partial}{\partial V} \left(\sum_i n_i \epsilon_i \right)$$

Radiation

$$\epsilon = h c k$$

Maxive particles

$$\epsilon = \frac{h^2 k^2}{2m}$$

Quantisation of k in box of side L

$$k = \frac{2\pi}{L} \cdot n = 2\pi n \cdot V^{-1/3}$$

$$\epsilon = \text{const} \times V^{-1/3}$$

$$\epsilon = \text{const} \times V^{-2/3}$$

$$\frac{\partial \epsilon}{\partial V} = - \frac{1}{3} \left(\frac{\epsilon}{V} \right)$$

$$\frac{\partial \epsilon}{\partial V} = - \frac{2}{3} \left(\frac{\epsilon}{V} \right)$$

$$p = - \sum n_i \left(\frac{\partial \epsilon_i}{\partial V} \right)$$

$$p = + \frac{1}{3} \frac{U}{V}$$

pressure of photons
in box

$$p = \frac{2}{3} \frac{U}{V}$$

pressure of particles
in box.

(121)

Phonons.

In a solid vibrating atoms interact strongly but can consider atomic motions as superposition of quantised lattice waves (phonons) which are weakly interacting

Lattice waves are bosons with distribution

function
$$f(\epsilon) = \frac{1}{\exp(\epsilon/kT) - 1}$$

Points

(i) For phonons degeneracy $g = 3$

representing transverse polarisations \uparrow and \leftrightarrow

and longitudinal waves \longleftrightarrow

(ii) Dispersion relation ϵ vs k

Depends on solid - monatomic
diatomic etc

but reasonable approximation at low ϵ
and k is

$$\epsilon = \frac{\hbar k}{2\pi} c_s \quad c_s = \text{velocity of sound (phonons)}$$

(iii) For N atoms.

$3N$ vibrations in 3D.

Gives rise to just $3N$ quantised
lattice waves (phonons).

Proper selection of waves needs details of
solid but reasonable approx —

Debye theory cuts off phonon

density of states at freq ν_D where

$$3N = \int_0^{\nu_D} g(\nu) d\nu = \int_0^{\nu_D} \frac{12\pi V}{c_s^3} \nu^2 d\nu$$

ν_D = Debye frequency

equivalent temp $\Theta_D = \frac{\hbar \nu_D}{k}$ (Debye temp)

(122a)

Debye theory vibration energy

$$U = \int_0^{\nu_D} g(\nu) d\nu \cdot h\nu \cdot f(\nu)$$

\downarrow density states $\nu \rightarrow \nu + d\nu$ \downarrow phonon energy \swarrow prob state with freq ν is occupied

$$U = \int_0^{\nu_D} \frac{12\pi V \nu^2 d\nu}{c_s^3} \cdot h\nu \cdot \frac{1}{[\exp(h\nu/kT) - 1]}$$

put $y = \frac{h\nu}{kT}$ $y_D = \frac{h\nu_D}{kT}$

$$\nu = \frac{kT}{h} y$$

$$d\nu = \left(\frac{kT}{h}\right) dy$$

Substituting - get.

(v) Debye theory vibration energy (see sheet)

$$U = V \cdot \left(\frac{12\pi h}{c^3} \right) \left(\frac{kT}{h} \right)^4 \int_0^{y_D} \frac{y^3 dy}{[\exp(y) - 1]}$$

$$\text{where } y = \frac{h\nu}{kT}$$

$$\text{and } y_D = \frac{h\nu_D}{kT}$$

For high temperature $T \gg \theta_D$

$$y \ll 1$$

$$\exp(y) \sim 1 + y \dots$$

$$\int \rightarrow \int_0^{y_D} y^2 dy = \frac{1}{3} \left(\frac{h\nu_D}{kT} \right)^3$$

$$\text{But from } 3N = \int_0^{\nu_D} \frac{12\pi V}{c^3} \nu^2 d\nu$$

$$\text{get } \nu_D^3 = \frac{3}{4\pi} \left(\frac{N}{V} \right) c^3$$

$$\text{Gives } U = 3NkT.$$

(124)

At low temperature $T \ll \Theta_D$

$$y_D \rightarrow \infty$$

$$\int_0^{\infty} \frac{y^3 dy}{\exp(y) - 1} \rightarrow \frac{\pi^4}{15}$$

$$U = \frac{12\pi h V}{c_s^3} \left(\frac{kT}{h} \right)^4 \cdot \frac{\pi^4}{15}$$

Express this in terms of $\Theta_D = \frac{h\nu_D}{k}$

$$\text{Via } c_s^3 = \frac{4\pi}{3} \left(\frac{V}{N} \right) \nu_D^3 \quad (\text{from above})$$

$$c_s^3 = \frac{4\pi}{3} \left(\frac{V}{N} \right) \left(\frac{k\Theta_D}{h} \right)^3$$

Substituting in expression for U gives

$$U = \frac{9\pi^4}{15} \cdot Nk \frac{T^4}{\Theta_D^3}$$

(125)

Heat capacity C_v .

$$C_v = \left(\frac{dU}{dT} \right)_v$$

At high temperature $C_v \rightarrow 3Nk$

At low temperature

$$C_v = \frac{d}{dT} \left(\frac{9\pi^4}{15} Nk \frac{T^4}{\Theta_D^3} \right)$$

$$C_v = \frac{12\pi^4}{5} Nk \left(\frac{T}{\Theta_D} \right)^3$$

These predictions of Debye theory agree well with experiment.

At low temps Debye prediction

$$C_v \propto T^3$$

much better than Einstein model prediction

$$C_v \propto \exp\left(-\frac{\Theta_E}{T}\right)$$

Conduction electrons.

How well does Fermi Dirac gas explain properties of conduction electrons in metals?

Treats correctly

- (i) Quantum energy states of particles in a box
- (ii) Fermion statistics
$$f(\epsilon) = \frac{1}{\exp(\frac{\epsilon - \mu}{kT}) + 1}$$

Problem

Fermi-Dirac gas assumes weak interactions between particles.

Actually Coulomb interactions of electrons

- (i) with the ions of lattice
- (ii) with other electrons

have energies $\approx \mu$ (Fermi energy) - strong interactions

Successes of F.D gas theory

Predicts correctly

- (i) Size of Fermi energy $\mu \sim$ few e.v.
- (ii) Heat capacity of conduction electrons

$$C_V = \gamma T$$

- (iii) Magnetic susceptibility of conduction electrons

χ is (i) small
(ii) independent of T

Incorrect details

- (i) Exact values of μ
- (ii) In $C_V = \gamma T$

actual values of γ greater than F.D prediction

for	Na	by factor	1.2
	Al	" "	1.5
	Pb	" "	2.1

Inaccuracies of F.D gas predictions arise because of neglect of interactions

Surprise is that it does so well.

Reasons why strong interactions cause only small effects.

(i) Long range cancellation.

Metal is charge neutral.

At distance $r >$ few lattice spacings
interaction of electron with

(i) the ions on lattice

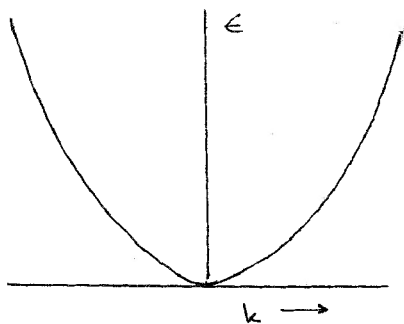
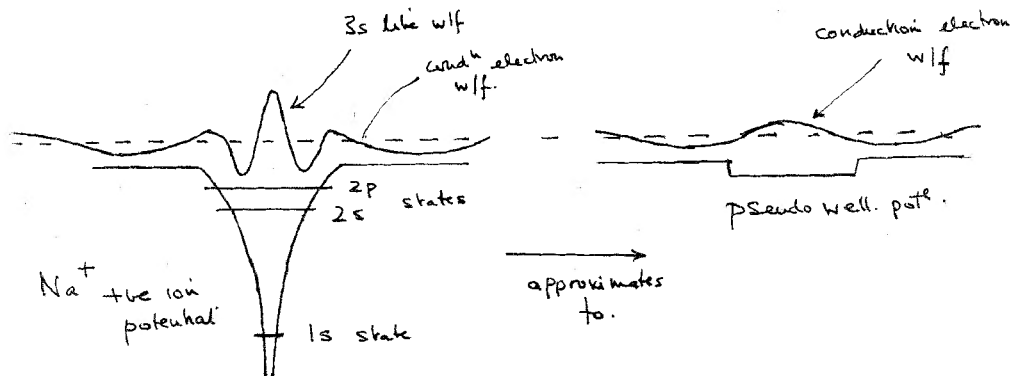
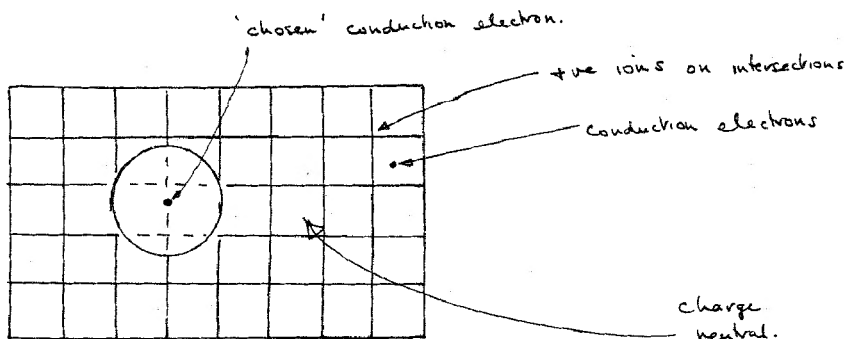
(ii) other conduction electrons

cancel out. — see picture on sheet.

(ii) Pauli principle — pseudo potential model.

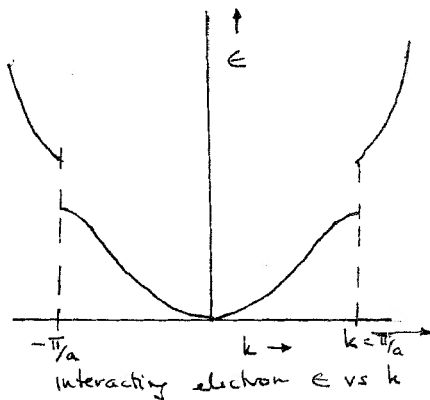
Electron close to the ion — sees it
as deep potential well.

Conduction electron interactions



free electron ϵ vs k

$$\epsilon = \frac{\hbar^2 k^2}{2m}$$



Example Na metal.

Conduction electron feels pot^l well of Na^+ ion

Na^+ ion has bound state electrons $(1s)^2(2s)^2(2p)^6$

Pauli principle requires condⁿ electron w/f to be orthogonal to all bound state w/fs.

Result — condⁿ electron w/f similar to 3s w/f (the ionised state)

This 3s w/f has high kinetic energy

High k.e in deep potential well — can approximate to

Smaller k.e and shallow well

— see picture on sheet.

Approximation close to weakly interacting electron.

(130)

Taking account of interactions

(1) Single particle approximation

Keep single particle theory

Effect of interactions change particle characteristics.

(Particle + effect of interactions) — called
quasiparticle

Effect of interactions felt through ϵ vs k
relation

Example.

$$\text{Mass of quasiparticle } m^* = \frac{\hbar^2}{\left(\frac{d^2\epsilon}{dk^2}\right)}$$

$$\text{For free particle } \epsilon = \frac{\hbar^2 k^2}{2m}$$

$$\text{Then } m^* = m$$

Quasiparticle mass can vary with

(i) particle energy